STABLE SETS AND MEAN LI-YORKE CHAOS IN POSITIVE ENTROPY SYSTEMS

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ABSTRACT. It is shown that in a topological dynamical system with positive entropy, there is a measure-theoretically "rather big" set such that a multivariant version of mean Li-Yorke chaos happens on the closure of the stable or unstable set of any point from the set. It is also proved that the intersections of the sets of asymptotic tuples and mean Li-Yorke tuples with the set of topological entropy tuples are dense in the set of topological entropy tuples respectively.

1. INTRODUCTION

Throughout this paper, by a *topological dynamical system* (X,T) (t.d.s. for short) we mean a compact metric space X with a homeomorphism T from X onto itself. The metric on X is denoted by d. For a t.d.s. (X,T), the stable set of a point $x\in X$ is defined as

$$W^s(x,T)=\{y\in X:\lim_{n\to+\infty} d(T^nx,T^ny)=0\}$$

and the unstable set of x is defined as

$$W^u(x,T)=\{y\in X:\lim_{n\to+\infty} d(T^{-n}x,T^{-n}y)=0\}.$$

Clearly, $W^s(x,T)=W^u(x,T^{-1})$ and $W^u(x,T)=W^s(x,T^{-1})$ for each $x\in X$. Stable and unstable sets play a big role in the study of smooth dynamical systems. Recently, there are many results related to the chaotic behavior and stable (unstable) sets in positive entropy systems (cf. [4, 6, 7, 9, 10, 12, 16]). So it indicates that stable and unstable sets are also important in the study of topological dynamical systems.

The chaotic behavior of a t.d.s. reflects the complexity of a t.d.s. Among the various definitions of chaos, Devaney's chaos, Li-Yorke chaos, positive entropy and the distributional chaos are the most popular ones. The implication among them have attracted a lot of attention. It was shown by Huang and Ye [17] that Devaney's chaos implies Li-Yorke one by proving that a non-periodic transitive t.d.s. with a periodic point is chaotic in the sense of Li and Yorke. In [4], Blanchard, Glasner, Kolyada and Maass proved that positive entropy also implies Li-Yorke chaos, that is if a t.d.s. (X,T) has positive entropy then there exists an uncountable subset S of X such that for any two distinct points $x,y\in S$,

$$\liminf_{n\to+\infty} d(T^nx,T^ny)=0 \text{ and } \limsup_{n\to+\infty} d(T^nx,T^ny)>0.$$

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We remark that the authors obtained the result using ergodic method, and for a combinatorial proof see [19]. Moreover, the result also holds for sofic group actions by Kerr and Li [20, Corollary 8.4].

In [6] Blanchard, Host and Ruette investigated the question if positive entropy implies the existence of non-diagonal asymptotic pairs. Among other things the authors showed that for positive entropy systems, many stable sets are not stable under T^{-1}. More precisely, if a T-invariant ergodic measure μ has positive entropy, then there exists $\eta > 0$ such that for μ-a.e. $x \in X$, one can find an uncountable subset F_x of $W^s(x,T)$ satisfying that for any $y \in F_x$,

$$\liminf_{n\to+\infty} d(T^{-n}x,T^{-n}y) = 0 \text{ and } \limsup_{n\to+\infty} d(T^{-n}x,T^{-n}y) \geq \eta.$$

Distributional chaos was introduced in [28], and there are at least three versions of distributional chaos in the literature (DC1, DC2 and DC3), see [2] for the details. It was known that positive entropy does not imply DC1 chaos [27]. In [29] Smítal conjectured that positive entropy implies DC2 chaos, and Oprocha showed that this conjecture holds for minimal uniformly positive entropy systems [23]. Very recently, Downarowicz [9] proved that positive entropy indeed implies DC2 chaos. Precisely, if a t.d.s. (X,T) has positive entropy then there exists an uncountable subset S of X such that for any two distinct points $x,y \in S$,

$$\liminf_{N\to+\infty} \frac{1}{N}\sum_{i=1}^{N} d(T^ix,T^iy) = 0 \text{ and } \limsup_{N\to+\infty} \frac{1}{N}\sum_{i=1}^{N} d(T^ix,T^iy) > 0.$$

Entropy pairs were introduced and studied in [3] by Blanchard, and the notion was extended to entropy tuples in [18] by Huang and Ye. Nowadays, the idea of using various tuples to obtain dynamical properties is widely accepted and developed, see for example the surveys [15, 24]. In [30], Xiong extended Huang-Ye's result related to Li-Yorke chaos to the multivariant version of Li-Yorke chaos. That is, if (X,T) is a non-periodic transitive system with a fixed point, then there exists a dense Mycielski subset (countable union of Cantor sets) S of X such that for any $n \geq 2$ and any n distinct points $x_1,\dots,x_n \in S$, one has

$$\liminf_{k\to+\infty} \max_{1\leq i<j\leq n} d\big(T^kx_i,T^kx_j\big) = 0 \text{ and } \limsup_{k\to+\infty} \min_{1\leq i<j\leq n} d\big(T^kx_i,T^kx_j\big) > 0.$$

Motivated by the above ideas, results and also by the consideration of Ornstein and Weiss in [25] (where they defined the notion of mean distality) we introduce the following multivariant version of mean Li-Yorke chaos.

A tuple $(x_1,\dots,x_n) \in X^{(n)}$ is called *mean Li-Yorke n-scrambled (with modulus η)* if

$$\liminf_{N\to+\infty} \frac{1}{N}\sum_{k=1}^{N} \max_{1\leq i<j\leq n} d\big(T^kx_i,T^kx_j\big) = 0$$

and

$$\limsup_{N\to+\infty} \frac{1}{N}\sum_{k=1}^{N} \min_{1\leq i<j\leq n} d\big(T^kx_i,T^kx_j\big) \geq \eta > 0.$$

A subset S of X is called *mean Li-Yorke n-scrambled (with modulus η)* if any n distinct points in S form a mean Li-Yorke n-scrambled tuple (with modulus η). The system (X,T) is called *mean Li-Yorke n-chaotic* if there is an uncountable mean Li-Yorke n-scrambled set.

The aim of this paper is to investigate the mean Li-Yorke chaos appearing in the closure of stable or unstable sets of a dynamical system with positive entropy, and the relationship among the sets of asymptotic tuples, mean Li-Yorke tuples and topological entropy tuples. More specifically, we have the following main results.

Theorem 1.1. *Let (X,T) be a t.d.s. with an ergodic invariant measure μ of positive entropy. Then there exists a sequence of positive numbers $\{\eta_n\}_{n=2}^{+\infty}$ satisfying that for μ-a.e. $x\in X$, there exists a Mycielski subset $K_x \subseteq \overline{W^s(x,T)}\cap\overline{W^u(x,T)}$ such that for every $n\geq 2$, K_x is a mean Li-Yorke n-scrambled set with modulus η_n both for T and for T^{-1}.*

It is observed in [9] that a pair is DC2-scrambled if and only if it is mean Li-Yorke 2-scrambled in our sense. We remark that the key tool in the proof of Theorem 1.1 is the excellent partition constructed in [6, Lemma 4] which is different from the one used in [9]. So among other things for $n=2$ we also obtain a new proof of Downarowicz's result in [9].

Theorem 1.2. *Let (X,T) be a t.d.s. with positive entropy. Then for any $n\geq 2$*

(1) *The intersection of the set of asymptotic n-tuples with the set of topological entropy n-tuples is dense in the set of topological entropy n-tuples.*
(2) *The intersection of the set of mean Li-Yorke n-tuples with the set of topological entropy n-tuples is dense in the set of topological entropy n-tuples.*

Note that in [6] and [13] the authors showed that the closure of the set of asymptotic pairs contains the set of entropy pairs, and the intersection of the set of proximal pairs with the set of entropy pairs is dense in the set of entropy pairs respectively. Thus Theorem 1.2 extends the mentioned results in [6] and [13].

Note also that the main results also hold for continuous maps with obvious modification, see Theorem 3.8 and Corollary 3.9.

The paper is organized as follows. In section 2, some backgrounds in ergodic theory and topological dynamics are introduced. In section 3, using an ad hoc excellent partition, the main result is proved. As corollaries, it is shown that each $n\geq 2$ the intersection of the set of asymptotic n-tuples with the set of topological entropy n-tuples, and the intersection of the set mean Li-Yorke n-tuples with the set of topological entropy n-tuples are dense in the set of topological entropy n-tuples respectively. Moreover, the main result of the paper is generalized to the non-invertible case.

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2. Preliminary

For a t.d.s. (X,T), denote by $\mathscr{B}_X$ the σ-algebra of Borel subsets of X. A *cover* of X is a family of Borel subsets of X whose union is X. An *open cover* is the one consisting of open sets. A *partition* of X is a cover of X by pairwise disjoint sets. Given a partition α of X and $x\in X$, denote by $\alpha(x)$ the atom of α containing x.

We denote the collection of finite partitions, finite covers and finite open covers of X by $\mathscr{P}_X$, $\mathscr{C}_X$ and $\mathscr{C}_X^o$, respectively. Let $\mathscr{U}$ and $\mathscr{V}$ be two covers. Their *join* $\mathscr{U}\bigvee\mathscr{V}$ is the cover $\{U\cap V : U\in\mathscr{U}, V\in\mathscr{V}\}$. For $\mathscr{U}\in\mathscr{C}_X$, we define $N(\mathscr{U})$ as the minimum among the cardinalities of the subcovers of $\mathscr{U}$. The *topological entropy* of $\mathscr{U}$ with respect to

T is

$$h_{\mathrm{top}}(T,\mathscr{U})=\lim_{N\to+\infty}\frac{1}{N}\log N\left(\bigvee_{i=0}^{N-1}T^{-i}\mathscr{U}\right).$$

The *topological entropy* of (X,T) is defined by $h_{\mathrm{top}}(T)=\sup_{\mathscr{U}\in\mathscr{C}_X^o}h_{\mathrm{top}}(T,\mathscr{U})$.

Let $\mathscr{M}(X)$, $\mathscr{M}(X,T)$ and $\mathscr{M}^e(X,T)$ be the collections of all Borel probability measures, T-invariant Borel probability measures and T-invariant ergodic measures on X, respectively. Then $\mathscr{M}(X)$ and $\mathscr{M}(X,T)$ are convex, compact metric spaces when endowed with the weak*-topology.

For any given $\alpha\in\mathscr{P}_X$ and $\mu\in\mathscr{M}(X)$, let $H_\mu(\alpha)=\sum_{A\in\alpha}-\mu(A)\log\mu(A)$. When $\mu\in\mathscr{M}(X,T)$, we define the *topological entropy* of α with respect to μ as

$$h_\mu(T,\alpha)=\lim_{n\to\infty}\frac{1}{n}H_\mu\left(\bigvee_{i=0}^{n-1}T^{-i}\alpha\right).$$

The *measure-theoretic entropy* of μ is defined by $h_\mu(T)=\sup_{\alpha\in\mathscr{P}_X}h_\mu(T,\alpha)$.

The relation between topological entropy and measure-theoretic entropy is the following well known variational principle: $h_{\mathrm{top}}(T)=\sup_{\mu\in\mathscr{M}^e(X,T)}h_\mu(T)$.

Let (X,T) be a t.d.s., $\mu\in\mathscr{M}(X,T)$ and $\mathscr{B}_\mu$ be the completion of $\mathscr{B}_X$ under the measure μ. Then $(X,\mathscr{B}_\mu,\mu,T)$ is a Lebesgue system. If $\{\alpha_i\}_{i\in I}$ is a countable family of finite partitions of X, the partition $\alpha=\bigvee_{i\in I}\alpha_i$ is called a *measurable partition*. The sets $A\in\mathscr{B}_\mu$, which are unions of atoms of α, form a sub-σ-algebra $\mathscr{B}_\mu$ denoted by $\widehat{\alpha}$ or α if there is no ambiguity. Every sub-σ-algebra of $\mathscr{B}_\mu$ coincides with a σ-algebra constructed in this way (mod μ).

For a measurable partition α, put $\alpha^-=\bigvee_{n=1}^{+\infty}T^{-n}\alpha$ and $\alpha^T=\bigvee_{n=-\infty}^{+\infty}T^{-n}\alpha$. Define in the same way $\mathscr{F}^-$ and $\mathscr{F}^T$ if $\mathscr{F}$ is a sub-σ-algebra of $\mathscr{B}_\mu$. It is clear that for a measurable partition α of X, $\widehat{\alpha^-}=(\widehat{\alpha})^-$ and $\widehat{\alpha^T}=(\widehat{\alpha})^T$(mod μ).

Let $\mathscr{F}$ be a sub-σ-algebra of $\mathscr{B}_\mu$ and α be a measurable partition of X with $\widehat{\alpha}=\mathscr{F}$ (mod μ). Then μ can be disintegrated over $\mathscr{F}$ as

$$\mu=\int_X\mu_x d\mu(x),$$

where $\mu_x\in\mathscr{M}(X)$ and $\mu_x(\alpha(x))=1$ for μ-a.e. $x\in X$. The disintegration can be characterized by the properties (2.1) and (2.2) as following:

$$\begin{aligned}&\text{for every } f\in L^1(X,\mathscr{B}_X,\mu),\ f\in L^1(X,\mathscr{B}_X,\mu_x) \text{ for } \mu\text{-a.e. } x\in X,\\ &\text{and the map } x\mapsto\int_X f(y)\,d\mu_x(y) \text{ is in } L^1(X,\mathscr{F},\mu);\end{aligned}\tag{2.1}$$

$$\text{for every } f\in L^1(X,\mathscr{B}_X,\mu),\ \mathbb{E}_\mu(f|\mathscr{F})(x)=\int_X f\,d\mu_x \text{ for } \mu\text{-a.e. } x\in X.\tag{2.2}$$

Then for any $f\in L^1(X,\mathscr{B}_X,\mu)$, one has

$$\int_X\left(\int_X f(y)\,d\mu_x(y)\right)d\mu(x)=\int_X f\,d\mu.$$

The support of $\mu \in \mathscr{M}(X)$ is defined to be the set of all points x in X for which every open neighborhood U of x has positive measure, that is

$$\begin{aligned}\mathrm{supp}(\mu) &= \{x \in X : \ \mu(U) > 0 \text{ for every open neighborhood } U \text{ of } x\} \\ &= X \setminus \bigcup \{U \subset X : \ U \text{ is open and } \mu(U) = 0\}.\end{aligned}$$

To prepare the proof of the main result in the next section we state or prove some useful lemmas in this section. The first one is the following.

Lemma 2.1. *Let X be a compact metric space, $\mu \in \mathscr{M}(X)$ and $\mathscr{B}_\mu$ be the completion of $\mathscr{B}_X$ under μ. If $\mathscr{F}_1, \mathscr{F}_2$ are two sub-σ-algebra of $\mathscr{B}_\mu$ with $\mathscr{F}_1 \supseteq \mathscr{F}_2$ and*

$$\mu = \int_X \mu_{i,x} d\mu(x)$$

is the disintegration of μ over $\mathscr{F}_i$ for $i = 1,2$, then $\mathrm{supp}(\mu_{1,x}) \subseteq \mathrm{supp}(\mu_{2,x})$ *for μ-a.e $x \in X$.*

Proof. First, choose $X_0 \in \mathscr{B}_\mu$ with $\mu(X_0) = 1$ such that $\mu_{1,x}, \mu_{2,x} \in \mathscr{M}(X)$ are well defined for all $x \in X_0$. Since X is a compact metric space, there exists a countable base Γ of the topology of X. For each $U \in \Gamma$ and $i = 1,2$, put

$$E_i(U) = \{x \in X_0 : \mu_{i,x}(U) = 0\}.$$

By (2.1) and (2.2), we have $E_i(U) \in \mathscr{B}_\mu$ for $i = 1,2$.

Next we are going to show $\mu(E_2(U) \setminus E_1(U)) = 0$ for every $U \in \Gamma$. Now fix $U \in \Gamma$. By (2.2), there exists $\widetilde{E_2(U)} \in \mathscr{F}_2$ with $\mu(\widetilde{E_2(U)} \Delta E_2(U)) = 0$. Moreover, combining this with (2.1) and (2.2), one has

$$\begin{aligned}\int_{E_2(U)} \mu_{1,x}(U) d\mu(x) &= \int_{\widetilde{E_2(U)}} \mu_{1,x}(U) d\mu(x) = \int_{\widetilde{E_2(U)}} \mathbb{E}_\mu(1_U | \mathscr{F}_1)(x) d\mu(x) \\ &= \int_{\widetilde{E_2(U)}} \mathbb{E}_\mu\big(\mathbb{E}_\mu(1_U | \mathscr{F}_1) | \mathscr{F}_2\big)(x) d\mu(x) \\ &= \int_{\widetilde{E_2(U)}} \mathbb{E}_\mu(1_U | \mathscr{F}_2)(x) d\mu(x) = \int_{\widetilde{E_2(U)}} \mu_{2,x}(U) d\mu(x) \\ &= \int_{E_2(U)} \mu_{2,x}(U) d\mu(x) = 0.\end{aligned}$$

This implies that $\mu(E_2(U) \setminus E_1(U)) = 0$.

Finally, we let $X_1 = X_0 \setminus \bigcup_{U \in \Gamma} \big(E_2(U) \setminus E_1(U)\big)$. Then $\mu(X_1) = 1$, and it is clear that for any $x \in X_1$ and $U \in \Gamma$, $x \in E_2(U)$ implies $x \in E_1(U)$. Thus for any $x \in X_1$,

$$\mathrm{supp}(\mu_{2,x}) = X \setminus \bigcup_{U \in \Gamma,\ x \in E_2(U)} U \supseteq X \setminus \bigcup_{U \in \Gamma,\ x \in E_1(U)} U = \mathrm{supp}(\mu_{1,x}).$$

This completes the proof. □

To state the next lemma which is the Lemma 4 in [6] we need some notions. Let (X,T) be a t.d.s., $\mu \in \mathscr{M}(X,T)$ and $\mathscr{B}_\mu$ be the completion of $\mathscr{B}_X$ under μ. The *Pinsker σ-algebra* $P_\mu(T)$ is defined as the smallest sub-σ-algebra of $\mathscr{B}_\mu$ containing $\{\xi \in \mathscr{P}_X : h_\mu(T,\xi) = 0\}$. It is well known that $P_\mu(T) = P_\mu(T^{-1})$ and $P_\mu(T)$ is T-invariant, i.e. $T^{-1}P_\mu(T) = P_\mu(T)$.

Lemma 2.2. *Let (X,T) be a t.d.s. and $\mu \in \mathscr{M}^e(X,T)$. Then there exists a sequence of partitions $\{W_i\}_{i=1}^{+\infty}$ in $\mathscr{P}_X$ and $0 = k_1 < k_2 < \cdots$ such that*

(1) $\lim\limits_{i\to+\infty} diam(W_i)=0$,

(2) $\lim\limits_{n\to+\infty} H_\mu(P_n|\mathscr{P}^-)=h_\mu(T)$, *where* $P_n=\bigvee\limits_{i=1}^{n} T^{-k_i}W_i$ *and* $\mathscr{P}=\bigvee_{n=1}^{\infty} P_n$.

(3) $\bigcap\limits_{n=0}^{+\infty} \widehat{T^{-n}\mathscr{P}^-}=P_\mu(T)$.

(4) $(T^{-n}\mathscr{P}^-)(x)\subseteq W^s(x,T)$ *for each* $n\in\mathbb{N}\cup\{0\}$ *and* $x\in X$, *where* $(T^{-n}\mathscr{P}^-)(x)$ *is the atom of* $T^{-n}\mathscr{P}^-$ *containing x.*

Proof. The proof of the lemma follows directly from that of Lemma 4 in [6]. For completeness, we outline the construction of $\{W_i\}_{i=1}^{+\infty}\subset\mathscr{P}_X$ and $0=k_1<k_2<\cdots$. Let $\{W_i\}_{i=1}^{+\infty}$ be an increasing sequence of finite partitions of X such that $\lim\limits_{i\to+\infty}\mathrm{diam}(W_i)=0$. Take $k_1=0$, then we find inductively $k_1,k_2,\cdots$ such that for each $m\geq 2$, one has

$$H_\mu(P_n|P_{m-1}^-)-H_\mu(P_n|P_m^-)<\frac{1}{n}\frac{1}{2^{m-n}}, n=1,2,\cdots,m-1,$$

where $P_j=\bigvee_{i=1}^{j} T^{-k_i}W_i$. It is not hard to check that (1)–(4) hold (see for example [26] or [14]). $\square$

We remark that the partition $\mathscr{P}$ constructed in Lemma 2.2 is an excellent partition (see [6] or [26]), which is a key tool in the proof of our main result.

Let (X,T) be a t.d.s. and $n\geq 2$. The n-th fold product system of (X,T) is denoted by $(X^{(n)},T^{(n)})$, where $X^{(n)}=X\times X\times\cdots\times X$ (n-times) and $T^{(n)}=T\times T\times\cdots\times T$ (n-times). And we set the diagonal of $X^{(n)}$ as $\Delta_n=\{(x,x,\ldots,x)\in X^{(n)}: x\in X\}$, and set $\Delta^{(n)}=\{(x_1,x_2,\ldots,x_n)\in X^{(n)}:$ there exist $1\leq i<j\leq n$ with $x_i=x_j\}$.

Let X be a metric space. A subset $K\subset X$ is called *a Mycielski set* if it is a union of countably many Cantor sets. This definition was introduced in [4]. Note that in [1] a Mycielski set is required to be dense. For convenience we restate here a version of Mycielski's theorem (see [22, Theorem 1]) which we shall use. See [1] for a comprehensive treatment of this topic.

Lemma 2.3 (Mycielski)**.** *Let X be a perfect compact metric space. Assume that for every $n\geq 2$, R_n is a dense G_δ subset of $X^{(n)}$. Then there exists a dense Mycielski subset K of X such that for every $n\geq 2$,*

$$K^{(n)}\subset R_n\cup\Delta^{(n)}.$$

Let (X,T) be a t.d.s., $n\geq 2$ and $\eta>0$. Recall that a tuple $(x_1,\ldots,x_n)\in X^{(n)}$ is *mean Li-Yorke n-scrambled with modulus* η if

$$\liminf_{N\to+\infty}\frac{1}{N}\sum_{k=1}^{N}\max_{1\leq i<j\leq n} d\big(T^kx_i,T^kx_j\big)=0$$

and

$$\limsup_{N\to+\infty}\frac{1}{N}\sum_{k=1}^{N}\min_{1\leq i<j\leq n} d\big(T^kx_i,T^kx_j\big)\geq\eta.$$

Denote by $MLY_{n,\eta}(X,T)$ the set of all mean Li-Yorke n-scrambled tuples with modulus η in (X,T), and $MLY_n(X,T)=\bigcup_{\eta>0}MLY_{n,\eta}(X,T)$.

Lemma 2.4. *Let (X,T) be a t.d.s., $n\geq 2$ and $\eta>0$. Then $MLY_{n,\eta}(X,T)$ is a G_δ subset of $X^{(n)}$.*

Proof. Let

$$P_n(X,T)=\bigcap_{m=1}^{+\infty}\bigcap_{\ell=1}^{+\infty}\Big(\bigcup_{N\geq\ell}\{(x_1,\dots,x_n)\in X^{(n)}:\frac{1}{N}\sum_{k=0}^{N-1}\max_{1\leq i<j\leq n}d\big(T^kx_i,T^kx_j)\big)<\tfrac{1}{m}\}\Big)$$

and

$$D_{n,\eta}(X,T)=\bigcap_{m=1}^{+\infty}\bigcap_{\ell=1}^{+\infty}\Big(\bigcup_{N\geq\ell}\{(x_1,\dots,x_n)\in X^{(n)}:\frac{1}{N}\sum_{k=0}^{N-1}\min_{1\leq i<j\leq n}d\big(T^kx_i,T^kx_j)\big)>\eta-\tfrac{1}{m}\}\Big).$$

It is easy to check that $P_n(X,T)$ and $D_{n,\eta}(X,T)$ are G_δ subsets of $X^{(n)}$. Then so is $MLY_{n,\eta}(X,T)$, since $MLY_{n,\eta}(X,T)=P_n(X,T)\bigcap D_{n,\eta}(X,T)$. $\square$

3. Proofs of the main results

In this section first we study the stable sets and unstable sets in positive systems, then give the proofs of the main results. Finally we show that our main results hold for continuous maps with obvious modification.

3.1. **Stable sets and unstable sets in positive systems.** First, we have the following lemma.

Lemma 3.1. *Let (X,T) be a t.d.s. and $\mu\in\mathscr{M}^e(X,T)$ with $h_\mu(T)>0$. If*

$$\mu=\int_X\mu_x d\mu(x)$$

is the disintegration of μ over the Pinsker σ-algebra $P_\mu(T)$, then for μ-a.e. $x\in X$,

$$\overline{W^s(x,T)\cap\operatorname{supp}(\mu_x)}=\operatorname{supp}(\mu_x)\text{ and }\overline{W^u(x,T)\cap\operatorname{supp}(\mu_x)}=\operatorname{supp}(\mu_x).$$

Proof. Since $P_\mu(T)$ is also the Pinsker σ-algebra of the system $(X,\mathscr{B}_\mu,\mu,T^{-1})$ and $W^s(x,T^{-1})=W^u(x,T)$, by symmetry it suffices to show that for μ-a.e. $x\in X$,

$$\overline{W^s(x,T)\cap\operatorname{supp}(\mu_x)}=\operatorname{supp}(\mu_x).$$

By Lemma 2.2, there exist $\{W_i\}_{i=1}^{+\infty}\subset\mathscr{P}_X$ and $0=k_1<k_2<\cdots$ satisfying that

(1) $\lim_{i\to+\infty}\operatorname{diam}(W_i)=0$.

(2) $\lim_{n\to+\infty}H_\mu(P_n|\mathscr{P}^-)=h_\mu(T)$, where $P_n=\bigvee_{i=1}^{n}T^{-k_i}W_i$ and $\mathscr{P}=\bigvee_{n=1}^\infty P_n$.

(3) $\bigcap_{n=0}^{+\infty}\widehat{T^{-n}\mathscr{P}^-}=P_\mu(T)$.

(4) $(T^{-n}\mathscr{P}^-)(x)\subseteq W^s(x,T)$ for each $n\in\mathbb{N}\cup\{0\}$ and $x\in X$.

For every $n\geq0$, let

$$\mu=\int_X\mu_{n,x}d\mu(x)$$

be the disintegration of μ over $\widehat{T^{-n}\mathscr{P}^-}$. Then for every $n\geq0$,

$$\mu_{n,x}((T^{-n}\mathscr{P}^-)(x))=1\text{ and so }\mu_{n,x}(W^s(x,T))=1\ \text{ for }\mu\text{-a.e. }x\in X.\tag{3.1}$$

Moreover, since

$$\widehat{\mathscr{P}^-}\supset\widehat{T^{-1}\mathscr{P}^-}\supset\widehat{T^{-2}\mathscr{P}^-}\supset\cdots\text{ and }\bigcap_{n=0}^{+\infty}\widehat{T^{-n}\mathscr{P}^-}=P_\mu(T),$$

there exists a set $X_1 \in \mathscr{B}_\mu$ with $\mu(X_1) = 1$ such that for any $x \in X_1$, $\lim_{n\to+\infty} \mu_{n,x} = \mu_x$ under the weak* topology (see for example [11, Corollary 5.21]), and for each $n \geq 0$,

$$\mu_{n,x}(W^s(x,T)) = 1 \text{ and } \operatorname{supp}(\mu_{n,x}) \subseteq \operatorname{supp}(\mu_{n+1,x}) \subseteq \operatorname{supp}(\mu_x) \tag{3.2}$$

by Lemma 2.1 and (3.1).

We will show that for every $x \in X_1$, $\operatorname{supp}(\mu_x) = \overline{W^s(x,T) \cap \operatorname{supp}(\mu_x)}$.

Fix $x \in X_1$. It is clear that $\overline{W^s(x,T) \cap \operatorname{supp}(\mu_x)} \subseteq \operatorname{supp}(\mu_x)$. By (3.2), one has that for any $n \geq 0$

$$\mu_{n,x}(\overline{W^s(x,T) \cap \operatorname{supp}(\mu_x)}) \geq \mu_{n,x}(W^s(x,T) \cap \operatorname{supp}(\mu_{n,x})) = 1.$$

Since $\lim_{n\to+\infty} \mu_{n,x} = \mu_x$ under the weak* topology,

$$\mu_x(\overline{W^s(X,T) \cap \operatorname{supp}(\mu_x)}) \geq \limsup_{n\to\infty} \mu_{n,x}(\overline{W^s(X,T) \cap \operatorname{supp}(\mu_x)}) = 1$$

and then $\operatorname{supp}(\mu_x) \subset \overline{W^s(x,T) \cap \operatorname{supp}(\mu_x)}$. □

Now we give the definition of entropy tuples introduced in [18], see [3] and [5] for the pair case respectively.

Definition 3.2. Let (X,T) be a t.d.s. and $n \geq 2$.

(1) A cover $\mathscr{U} = \{U_1, \dots, U_k\}$ of X is said to be *admissible* with respect to $(x_1, \dots, x_n) \in X^{(n)}$ if for each for $1 \leq i \leq k$ there exists j_i such that $x_{j_i} \notin \overline{U_i}$.

(2) A tuple $(x_1, \dots, x_n) \in X^{(n)}$ is called a *topological entropy n-tuple*, if at least two of the points in $(x_1, \dots, x_n)$ are distinct and any admissible open cover with respect to $(x_1, \dots, x_n)$ has positive topological entropy. Denote by $E_n(X,T)$ the set of all topological entropy n-tuples.

(3) A tuple $(x_1, \dots, x_n) \in X^{(n)}$ is called an *entropy n-tuple for* $\mu \in \mathscr{M}(X,T)$, if at least two of the points in $(x_1, \dots, x_n)$ are distinct and any admissible Borel partition α with respect to $(x_1, \dots, x_n)$ has positive measure-theoretic entropy. Denote by $E_n^\mu(X,T)$ the set of all entropy n-tuples for μ.

Let (X,T) be a t.d.s. and $\mu \in \mathscr{M}(X,T)$. For every $n \geq 2$, define a measure $\lambda_n(\mu)$ on $(X^{(n)}, T^{(n)})$ by letting

$$\lambda_n(\mu) = \int_X \mu_x^{(n)} d\mu(x),$$

where $\mu_x^{(n)} = \mu_x \times \mu_x \times \cdots \times \mu_x$ (n-times). It is well known that when μ is ergodic with positive entropy (see for example [4, 18]), μ_x is non-atomic for μ-a.e $x \in X$ and $\lambda_n(\mu)$ is a $T^{(n)}$-invariant ergodic measure on $X^{(n)}$. The relation between topological entropy tuples and entropy tuples for μ is the following.

Proposition 3.3 ([5, 13, 18]). *Let (X,T) be a t.d.s. and $\mu \in \mathscr{M}^e(X,T)$. Then for every $n \geq 2$,*

(1) $E_n^\mu(X,T) = \operatorname{supp}(\lambda_n(\mu)) \setminus \Delta_n$ *and*

(2) $E_n(X,T) = \overline{\bigcup_{\mu \in \mathscr{M}^e(X,T)} E_n^\mu(X,T)} \setminus \Delta_n$.

Let (X,T) be a t.d.s. and $n \geq 2$. A tuple $(x_1, \dots, x_n) \in X^{(n)}$ is called *n-asymptotic* if

$$\lim_{k\to+\infty} \max_{1\leq i<j\leq n} d(T^k x_i, T^k x_j) = 0.$$

The set of all asymptotic n-tuples is denoted by $Asy_n(X,T)$. It was shown in [17] that $Asy_2(X,T)$ is of first category if (X,T) is sensitive. It was known [13, 18] that if μ is ergodic with positive entropy then $(E_2^\mu(X,T)\cup\Delta_2, T\times T)$ is transitive and $\lambda_2(\mu)((E_2^\mu(X,T)\cup\Delta_2)\cap Asy_2(X,T))=0$ since an asymptotic pair can not be a transitive point of $(E_2^\mu(X,T)\cup\Delta_2, T\times T)$. Note that Theorem 1.2(1) states that for every $n\geq 2$, $Asy_n(X,T)\cap E_n(X,T)$ is dense in $E_n(X,T)$.

Proof of Theorem 1.2(1). Let μ be an ergodic invariant measure with positive entropy. Then by Lemma 3.1, there exists $X_1\in\mathscr{B}_X$ with $\mu(X_1)=1$ such that for each $x\in X_1$, $W^s(x,T)\cap\mathrm{supp}(\mu_x)$ is dense in $\mathrm{supp}(\mu_x)$. It is clear $W^s(x,T)^{(n)}\subset Asy_n(X,T)$ and thus $Asy_n(X,T)\cap\mathrm{supp}(\mu_x)^{(n)}\supset W^s(x,T)^{(n)}\cap\mathrm{supp}(\mu_x)^{(n)}\supset(W^s(x,T)\cap\mathrm{supp}(\mu_x))^{(n)}$ which implies that

$$\overline{Asy_n(X,T)\cap\mathrm{supp}(\mu_x)^{(n)}}=\mathrm{supp}(\mu_x)^{(n)}\text{ for each }x\in X_1. \tag{3.3}$$

By Proposition 3.3(1), $E_n^\mu(X,T)=\mathrm{supp}(\lambda_n(\mu))\setminus\Delta_n$, so we have $\lambda_n(\mu)(E_n^\mu(X,T)\cup\Delta_n)=1$. This implies that there exists $X_2\in\mathscr{B}_X$ with $\mu(X_2)=1$ such that for each $x\in X_2$, one has $\mu_x^{(n)}(E_n^\mu(X,T)\cup\Delta_n)=1$ by the definition of $\lambda_n(\mu)$. Thus, $\mathrm{supp}(\mu_x)^{(n)}=\mathrm{supp}\,\mu_x^{(n)}\subset E_n^\mu(X,T)\cup\Delta_n$ and hence $\mathrm{supp}(\mu_x)^{(n)}\setminus\Delta_n(X)\subset E_n^\mu(X,T)$. Now by (3.3) and the fact that μ_x is non-atomic we have for each $x\in X_1\cap X_2$

$$\overline{Asy_n(X,T)\cap E_n^\mu(X,T)}\supset\overline{Asy_n(X,T)\cap(\mathrm{supp}(\mu_x)^{(n)}\setminus\Delta_n(X))}=\mathrm{supp}(\mu_x)^{(n)}$$

which implies that

$$\overline{Asy_n(X,T)\cap E_n^\mu(X,T)}\supset\bigcup_{x\in X_1\cap X_2}\mathrm{supp}(\mu_x)^{(n)}.$$

Thus we get that

$$\lambda_n(\mu)(\overline{Asy_n(X,T)\cap E_n^\mu(X,T)})\geq\int_X\mu_x^{(n)}(\bigcup_{x\in X_1\cap X_2}\mathrm{supp}(\mu_x)^{(n)})d\mu(x)=1$$

which implies that

$$\overline{Asy_n(X,T)\cap E_n^\mu(X,T)}\supset\mathrm{supp}(\lambda_n(\mu))\supset E_n^\mu(X,T), \tag{3.4}$$

i.e. $Asy_n(X,T)\cap E_n^\mu(X,T)$ is dense in $E_n^\mu(X,T)$.

By Proposition 3.3(2) and (3.4), for each $\mu\in\mathscr{M}^e(X,T)$ we have

$$\overline{Asy_n(X,T)\cap E_n(X,T)}\supset\overline{Asy_n(X,T)\cap E_n^\mu(X,T)}=E_n^\mu(X,T)$$

which implies that

$$\overline{Asy_n(X,T)\cap E_n(X,T)}\supset\overline{\bigcup_{\mu\in\mathscr{M}^e(X,T)}E_n^\mu(X,T)}\supset E_n(X,T),$$

i.e. $Asy_n(X,T)\cap E_n(X,T)$ is dense in $E_n(X,T)$. □

Remark 3.4. *We have defined an asymptotic pair in the positive direction. It is clear that we may define a two-sided asymptotic pair (x,y) by $\lim_{n\to\infty}d(T^nx,T^ny)=0$. It was know that positive entropy does not implies the existence of a non-diagonal two-sided asymptotic pair, see* [21, Example 3.4] *by Lind and Schmidt. In* [8] *Chung and Li showed that if Γ is a polycyclic-by-finite group and Γ acts on a compact abelian*

group X expansively by automorphisms, then $\{x\in X:(x,e)\in Asy_2(X,\Gamma)\}$ is dense in $\{x\in X:(x,e)\in E_2(X,\Gamma)\}$, where e is the unit of Γ.

3.2. **Proofs of Theorems 1.1 and 1.2(2).**

Proof of Theorem 1.1. Let $\mathscr{B}_\mu$ be the completion of $\mathscr{B}_X$ under μ. Then $(X,\mathscr{B}_\mu,\mu,T)$ is a Lebesgue system. Let $P_\mu(T)$ be the Pinsker σ-algebra of $(X,\mathscr{B}_\mu,\mu,T)$. Let

$$\mu=\int_X \mu_x d\mu(x)$$

be the disintegration of μ over $P_\mu(T)$. By Lemma 3.1, there exists a set $X_1\in\mathscr{B}_\mu$ with $\mu(X_1)=1$ such that for any $x\in X_1$,

$$\overline{W^s(x,T)\cap\operatorname{supp}(\mu_x)}=\operatorname{supp}(\mu_x) \text{ and } \overline{W^s(x,T^{-1})\cap\operatorname{supp}(\mu_x)}=\operatorname{supp}(\mu_x). \tag{3.5}$$

For every $n\geq 2$, let $\lambda_n(\mu)$ be the measure on $(X^{(n)},T^{(n)})$ defined before. Recall that $\Delta^{(n)}=\{(x_1,x_2,\dots,x_n)\in X^{(n)}:$ there exist $1\leq i<j\leq n$ with $x_i=x_j\}$. Since μ_x is non-atomic for a.e. $x\in X$, by the Fubini's theorem, $\lambda_n(\mu)(\Delta^{(n)})=0$. It is easy to check that

$$X^{(n)}\setminus\Delta^{(n)}=\bigcup_{k=1}^\infty\{(x_1,\dots,x_n)\in X^{(n)}:\min_{1\leq i<j\leq n}d(x_i,x_j)>\tfrac{1}{k}\}.$$

Thus there exists $\tau>0$ such that $\lambda_n(\mu)(W_n)>0$, where

$$W_n=\{(x_1,\dots,x_n)\in X^{(n)}:\min_{1\leq i<j\leq n}d(x_i,x_j)>\tau\}.$$

Let $\eta_n=\tau\lambda_n(\mu)(W_n)$, and let G_n^+ be the set of all generic points of $\lambda_n(\mu)$ for $T^{(n)}$, that is $(x_1,\dots,x_n)\in G_n^+$ if and only if

$$\frac{1}{N}\sum_{i=0}^{N-1}\delta_{(T^{(n)})^i(x_1,\dots,x_n)}\to\lambda_n(\mu)$$

under the weak*-topology, where δ_y is the point mass on y. Then $\lambda_n(\mu)(G_n^+)=1$. For every $(x_1,\dots,x_n)\in G_n$,

$$\begin{aligned}\limsup_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\min_{1\leq i<j\leq n}d(T^kx_i,T^kx_j)&\geq\limsup_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\tau\delta_{(T^{(n)})^k(x_1,\dots,x_n)}(W_n)\\&\geq\tau\liminf_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\delta_{(T^{(n)})^k(x_1,\dots,x_n)}(W_n)\\&\geq\tau\lambda_n(\mu)(W_n)=\eta_n.\end{aligned}$$

This shows that $G_n^+\subset D_{n,\eta_n}(X,T)$, where D_{n,η_n} is defined in the proof of Lemma 2.4. Similarly, if we let G_n^- be the set of all generic points of $\lambda_n(\mu)$ for $(T^{-1})^{(n)}$ and then $\lambda_n(\mu)(G_n^-)=1$ and $G_n^-\subset D_{n,\eta_n}(X,T^{-1})$.

Since μ_x is non-atomic for μ-a.e. $x\in X$ and

$$1=\lambda_n(\mu)(G_n^+\cap G_n^-)=\int_X\mu_x^{(n)}(G_n^+\cap G_n^-)d\mu(x),$$

there exists a set $X_2\in\mathscr{B}_\mu$ with $\mu(X_2)=1$ such that μ_x is non-atomic and $\mu_x^{(n)}(G_n^+\cap G_n^-)=1$ for all $x\in X_2$ and all $n\geq 2$. Now let $X_0=X_1\cap X_2$. Then $\mu(X_0)=1$.

Now fix $x \in X_0$. Then $\mathrm{supp}(\mu_x)$ is a perfect closed subset of X, since μ_x is non-atomic. By the construction of X_2, we have $\mu_x^{(n)}(G_n^+ \cap G_n^- \cap \mathrm{supp}(\mu_x)^{(n)}) = 1$, and then $G_n^+ \cap G_n^- \cap \mathrm{supp}(\mu_x)^{(n)}$ is dense in $\mathrm{supp}(\mu_x)^{(n)}$. So

$$D_{n,\eta_n}(X,T) \cap D_{n,\eta_n}(X,T^{-1}) \cap \mathrm{supp}(\mu_x)^{(n)}$$

is a dense G_δ subset of $\mathrm{supp}(\mu_x)^{(n)}$ by Lemma 2.4, and the facts that $G_n^+ \subset D_{n,\eta_n}(X,T)$ and $G_n^- \subset D_{n,\eta_n}(X,T^{-1})$. By the construction of X_1, we have $Asy_n(X,T) \cap \mathrm{supp}(\mu_x)^{(n)}$ and $Asy_n(X,T^{-1}) \cap \mathrm{supp}(\mu_x)^{(n)}$ are dense in $\mathrm{supp}(\mu_x)^{(n)}$. So

$$P_n(X,T) \cap P_n(X,T^{-1}) \cap \mathrm{supp}(\mu_x)^{(n)}$$

is a dense G_δ subset of $\mathrm{supp}(\mu_x)^{(n)}$ by Lemma 2.4, and the facts that $Asy_n(X,T) \subset P_n(X,T)$ and $Asy_n(X,T^{-1}) \subset P_n(X,T^{-1})$. Again by Lemma 2.4, we have that

$$MLY_{n,\eta_n}(X,T) \cap MLY_{n,\eta_n}(X,T^{-1}) \cap \mathrm{supp}(\mu_x)^{(n)}$$

is a dense G_δ subset of $\mathrm{supp}(\mu_x)^{(n)}$. By Mycielski's theorem (Lemma 2.3), there exists a dense Mycielski subset K_x of $\mathrm{supp}(\mu_x)$ such that for every $n \geq 2$,

$$K_x^{(n)} \subset (MLY_{n,\eta_n}(X,T) \cap MLY_{n,\eta_n}(X,T^{-1})) \cup \Delta^{(n)}.$$

Then K_x is as required. □

Recall that Theorem 1.2(2) states that for every $n \geq 2$, $MLY_n(X,T) \cap E_n(X,T)$ is dense in $E_n(X,T)$.

Proof of Theorem 1.2(2). Let μ be an ergodic invariant measure with positive entropy. By the proof of Theorem 1.1 there exists $\eta_n > 0$ such that for μ-a.e. $x \in X$, $MLY_{n,\eta_n}(x,T) \cap \mathrm{supp}(\mu_x)^{(n)}$ is dense in $\mathrm{supp}(\mu_x)^{(n)}$. By the same argument as in the proof of Theorem 1.2(1), we get the result. □

Corollary 3.5. *Let (X,T) be a t.d.s. If there is an invariant measure μ of full support such that $(X,\mathscr{B}_X,\mu,T)$ is a Kolmogorov system, then there exists a dense Mycielski subset K of X such that for any $n \geq 2$, K is mean Li-Yorke n-scrambled (with modulus $\eta_n > 0$).*

Proof. If $(X,\mathscr{B}_X,\mu,T)$ is a Kolmogorov system, then the Pinsker σ-algebra $P_\mu(T) = \{\emptyset, X\}$ (mod μ). So the disintegration of μ over $P_\mu(T)$ is trivial, that is for μ-a.e. $x \in X$, $\mu_x = \mu$. Now the result follows from Theorem 1.1, since $\mathrm{supp}(\mu_x) = \mathrm{supp}(\mu) = X$. □

Recall that we say a t.d.s. (X,T) has *uniformly positive entropy* (see [3]) if any cover of X by two non-dense open sets has positive entropy. It is not hard to see that a t.d.s. (X,T) has uniformly positive entropy if and only if $E_2(X,T) \cup \Delta_2 = X^{(2)}$.

Corollary 3.6. *If a uniformly positive entropy system (X,T) admits an ergodic invariant measure μ with full support and $h_\mu(T) > 0$, then there exists a dense Mycielski subset K of X such that K is mean Li-Yorke 2-scrambled (with modulus $\eta > 0$).*

Proof. By Theorem1.2(2), there exists $\eta_0 > 0$ such that $MLY_{2,\eta_0}(X,T)$ is dense in $E_2^\mu(X,T)$. Since μ has full support, $\Delta_2 \subset \overline{E_2^\mu(X,T)} \subset \overline{MLY_{2,\eta_0}(X,T)}$. Let $\eta = \eta_0/2$.

First, we show that $D_{2,\eta}(X,T)$ is dense in $X^{(2)}$. For any $(x_1,x_2)\in X^{(2)}$ and $\varepsilon>0$, there exists $(x_1',x_1'')\in MLY_{2,\eta_0}(X,T)$ with $d(x_1,x_1')<\varepsilon$ and $d(x_1,x_1'')<\varepsilon$. Then

$$\begin{aligned}\eta_0 &\le \limsup_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1} d(T^k x_1',T^k x_1'')\\ &\le \limsup_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\big(d(T^k x_1',T^k x_2)+d(T^k x_1'',T^k x_2)\big)\\ &\le \limsup_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1} d(T^k x_1',T^k x_2)+\limsup_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1} d(T^k x_1'',T^k x_2)\end{aligned}$$

Then either $(x_1',x_2)\in D_{2,\eta}(X,T)$ or $(x_1'',x_2)\in D_{2,\eta}(X,T)$. This shows that $D_{2,\eta}(X,T)$ is dense in $X^{(2)}$.

Since (X,T) has uniformly positive entropy, by Theorem1.2(1), $Asy_2(X,T)$ is dense in $X^{(2)}$, and then $P_2(X,T)$ is also dense in $X^{(2)}$. Therefore, $MLY_{2,\eta}(X,T)=P_2(X,T)\cap D_{2,\eta}(X,T)$ is a dense G_δ subset of $X^{(2)}$. By Mycielski theorem, there exists a dense Mycielski subset K of X such that $K^2\subset MLY_{2,\eta}(X,T)\cup\Delta_2$. Then K is a mean Li-Yorke 2-scrambled set with modulus η. $\square$

3.3. **Non-invertible case.** In this subsection, we will generalize the main results to the non-invertible case. Let (X,T) be a non-invertible t.d.s., i.e. X is a compact metric space, and $T:X\to X$ is a continuous surjective map but not one-to-one.

For a t.d.s. (X,T) with metric d, we say that $(\widetilde{X},\widetilde{T})$ is the *natural extension* of (X,T), if $\widetilde{X}=\{(x_1,x_2,\cdots):T(x_{i+1})=x_i,\ i\in\mathbb{N}\}$ is a subspace of the product space $X^{\mathbb{N}}=\prod_{i=1}^{\infty}X$ endowed with the compatible metric $\tilde{d}$ defined by

$$\tilde{d}((x_1,x_2,\cdots),(y_1,y_2,\cdots))=\sum_{i=1}^{\infty}\frac{d(x_i,y_i)}{2^i},$$

and $\widetilde{T}:\widetilde{X}\to\widetilde{X}$ is the shift homeomorphism, i.e. $\widetilde{T}(x_1,x_2,\cdots)=(T(x_1),x_1,x_2,\cdots)$. Let $\pi:\widetilde{X}\to X$ be the projection to the first coordinate. Then $\pi:(\widetilde{X},\widetilde{T})\to(X,T)$ is a factor map. It is clear that a tuple $(\tilde{x}_1,\dots,\tilde{x}_n)$ is asymptotic in $\tilde{X}$ if and only if the tuple $(\pi(\tilde{x}_1),\dots,\pi(\tilde{x}_n))$ is asymptotic in X, that is for every $n\ge 2$, $\pi^{(n)}(Asy_n(\tilde{X},\tilde{T}))=Asy_n(X,T)$. Using the fact that for $\tilde{x},\tilde{y}\in\widetilde{X}$ and $k\ge 0$,

$$\begin{aligned}\sum_{p=1}^{k+1}\frac{d(T^{k-p+1}\pi(\tilde{x}),T^{k-p+1}\pi(\tilde{y}))}{2^p}&\le \tilde{d}(\widetilde{T}^k\tilde{x},T^k\tilde{y})\\ &\le \sum_{p=1}^{k+1}\frac{d(T^{k-p+1}\pi(\tilde{x}),T^{k-p+}\pi(\tilde{y}))}{2^p}+\frac{\operatorname{diam}(X)}{2^k},\end{aligned}$$

the following lemma is easy to verify.

Lemma 3.7. *For every $(\tilde{x}_1,\dots,\tilde{x}_n)\in\widetilde{X}^{(n)}$, one has*

$$\limsup_{N\to+\infty}\frac{1}{N}\sum_{k=0}^{N-1}\min_{1\le i<j\le n}\tilde{d}(\widetilde{T}^k\widetilde{x}_i,\widetilde{T}^k\widetilde{x}_j)=\limsup_{N\to+\infty}\frac{1}{N}\sum_{k=0}^{N-1}\min_{1\le i<j\le n}d(T^k\pi(\widetilde{x}_i),T^k\pi(\widetilde{x}_j))$$

and

$$\liminf_{N\to+\infty}\frac{1}{N}\sum_{k=0}^{N-1}\max_{1\le i<j\le n}\tilde{d}(\widetilde{T}^k\widetilde{x}_i,\widetilde{T}^k\widetilde{x}_j)=\liminf_{N\to+\infty}\frac{1}{N}\sum_{k=0}^{N-1}\max_{1\le i<j\le n}d(T^k\pi(\widetilde{x}_i),T^k\pi(\widetilde{x}_j)).$$

Thus we have

Theorem 3.8. *Let (X,T) be a non-invertible t.d.s. and $\mu\in\mathscr{M}^e(X,T)$ with $h_\mu(T)>0$. Then for each $k\ge 2$ there exists $\eta_k>0$ satisfying that for μ-a.e. $x\in X$, there exists a Mycielski subset $K_x\subseteq\overline{W^s(x,T)}$ such that for any $n\ge 2$, K is mean Li-Yorke n-scrambled with modulus η_n.*

Proof. Let $(\widetilde{X},\widetilde{T})$ be the natural extension of (X,T) and $\pi:\widetilde{X}\to X$ be the projection to the first coordinate. It is well known that there exists $\widetilde{\mu}\in\mathscr{M}^e(\widetilde{X},\widetilde{T})$ such that $\pi(\widetilde{\mu})=\mu$. Clearly, $h_{\widetilde{\mu}}(\widetilde{T})\ge h_\mu(T)>0$. By Theorem 1.1, there exists a Borel subset $\widetilde{X}_0\subseteq\widetilde{X}$ with $\widetilde{\mu}(\widetilde{X}_0)=1$ such that for any $\widetilde{x}\in\widetilde{T}_0$, there exists a Mycielski subset $K_{\widetilde{x}}$ of $\overline{W^s(\widetilde{x},\widetilde{T})}$ satisfying the conclusion of Theorem 1.1.

Let $X_0=\pi(\widetilde{X}_0)$. Then X_0 is a μ-measurable set and $\mu(X_0)=1$. For any $x\in X_0$, there exists $\widetilde{x}\in\widetilde{X}_0$ such that $\pi(\widetilde{x})=x$. Let $K_x=\pi(K_{\widetilde{x}})$. Then by Lemma 3.7, K_x satisfies our requirements. □

Corollary 3.9. *Let (X,T) be a non-invertible t.d.s. with positive entropy. Then for every $n\ge 2$, $Asy_n(X,T)\cap E_n(X,T)$ and $MLY_n\cap E_n(X,T)$ are dense in $E_n(X,T)$.*

Proof. Let $(\widetilde{X},\widetilde{T})$ be the natural extension of (X,T) and $\pi:\widetilde{X}\to X$ be the projection to the first coordinate. Then $Asy_n(X,T)=\pi^{(n)}(Asy_n(\widetilde{X},\widetilde{T}))$, $MLY_n(X,T)=\pi^{(n)}(MLY_n(\widetilde{X},\widetilde{T}))$ and $E_n(X,T)\subset\pi^{(n)}(E_n(\widetilde{X},\widetilde{T}))$ (see [18]). Thus, the result follows from Theorem 1.2. □

4. Final remarks

Downarowicz and Yacroix [10] showed that positive topological entropy implies DC$1\frac{1}{2}$, which is delicately weaker than DC1. We restate their result as follows.

Theorem 4.1 ([10])**.** *If a t.d.s. (X,T) has positive entropy, then there exists an uncountable subset S of X such that for any two distinct points $x,y\in S$, one has*

(1) *for every $t>0$, the set $\{k\in\mathbb{Z}_+:\ d(T^kx,T^ky)<t\}$ has upper density 1, and*
(2) *for every $s\in(0,1)$, there exists $t_s>0$ such that the upper density of the set $\{k\in\mathbb{Z}_+:\ d(T^kx,T^ky)>t_s\}$ is at least s.*

After a little modification of the proof of Theorem 1.1, we can strengthen their result as follows.

Theorem 4.2. *Let (X,T) be a t.d.s. If there exists an invariant ergodic measure μ with $h_\mu(T)>0$, then for μ-a.e. $x\in X$, there exists a Mycielski set $K_x\subseteq\overline{W^s(x,T)}\cap\overline{W^u(x,T)}$ such that for any $n\ge 2$ and any distinct n points $x_1,\dots,x_n\in K_x$, one has*

(1) *for every $t>0$, the set $\{k\in\mathbb{Z}_+:\ \max_{1\le i<j\le n}d(T^kx,T^ky)<t\}$ has upper density 1, and*
(2) *for every $s\in(0,1)$, there exists $t_{n,s}>0$ such that the upper density of the set $\{k\in\mathbb{Z}_+:\ \min_{1\le i<j\le n}d(T^kx,T^ky)>t_{n,s}\}$ is at least s.*

Sketch of the proof. We keep the notation as in the proof of Theorem 1.1. Since μ_x is atomless for a.e. $x \in X$, by the Fubini's theorem, $\lambda_n(\mu)(\Delta^{(n)}) = 0$ and then

$$\lim_{\varepsilon \to 0} \lambda_n(\mu)([\Delta^{(n)}]_\varepsilon) = 0,$$

where $[\Delta^{(n)}]_\varepsilon = \{(x_1, \dots, x_n) \in X^{(n)} : \max_{1 \le i < j \le n} d(x_i, x_j) \le \varepsilon\}$. For every every $s \in (0,1)$, there exists $t_{n,s} > 0$ such that $\lambda_n(\mu)([\Delta^{(n)}]_{t_{n,s}}) < 1 - s$. Then

$$\lambda_n(\mu)(X^{(n)} \setminus [\Delta^{(n)}]_{t_{n,s}}) > s.$$

For every $n \ge 2$, let A_n and B_n be the set of all points $(x_1, \dots, x_n) \in X^{(n)}$ satisfying the condition (1) and (2), respectively. Clearly, for every $n \ge 2$, $Asy_n(X,T) \subset A_n$ and $G_n^+ \subset B_n$. Then it is not hard to see that for every $n \ge 2$ and $x \in X_0$, $A_n \cap B_n \cap \mathrm{supp}(\mu_x)^{(n)}$ contains a dense G_δ subset of $\mathrm{supp}(\mu_x)^{(n)}$. By Mycielski's theorem, there exists a dense Mycielski subset K_x of $\mathrm{supp}(\mu_x)$ such that for every $n \ge 2$, $K_x^{(n)} \subset (A_n \cap B_n) \cup \Delta^{(n)}$. Then K_x is as required. □

Recall that in [19] Kerr and Li gave a combinatorial proof of the fact that positive entropy implies Li-Yorke's chaos. Since the proofs in [9] and in our paper are measure-theoretical, so finding a topological or combinatorial proof of the fact that positive entropy implies DC2 chaos or an uncountable mean Li-Yorke n-scrambled set for $n \ge 2$ is natural. Unfortunately, we do not have one at this moment.

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